An Exploration of the Geometry of Hyperbolic Maps

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Hyperbolic Numbers

The **hyperbolic** numbers are have the form $x + jy$ where x, y are real numbers and $j^2 = 1$. We let $\mathcal{H}$ denote the set of all hyperbolic numbers. If $a + jb, c + jd \in \mathcal{H}$ then

$$(a + jb)(c + jd) = ac + adj + jbc + j^2bd = ac + bd + j(ad + bc).$$

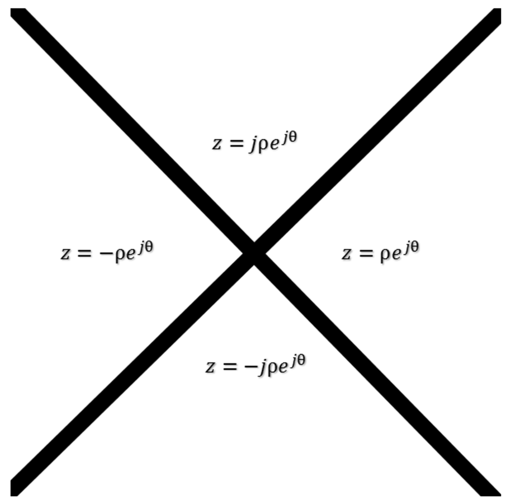
The map $\mathbf{M} : \mathcal{H} \to \mathrm{M}_{\mathcal{H}}$ defined below is an algebra isomorphism:

$$\mathbf{M}(a + bj) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Since $\det(A) = 0$ if and only if A^{-1} does not exist we find $(a + jb)^{-1}$ does not exist whenever $\det(\mathbf{M}(a + bj)) = a^2 - b^2 = 0$. These are the **zero-divisors:**

$$\mathbf{zd}(\mathcal{H}) = \{a + bj \mid a^2 = b^2\}$$

whereas the **units** are $\mathcal{H}^\times = \{a + bj \mid a^2 \neq b^2\}$.



We define the **hyperbolic power function** with the sectors above in mind:

$$z^c = \begin{cases} \pm\exp\left(c\mathrm{Log}(z)\right) & \text{when } z = \pm\rho e^{j\theta} \\ \pm j\exp\left(c\mathrm{Log}_j(z)\right) & \text{when } z = \pm j\rho e^{j\theta} \end{cases}$$

where we define

$$\mathrm{Log}(z) = \ln\sqrt{x^2 - y^2} + j\tanh^{-1}(y/x)$$
$$\mathrm{Log}(z) = \ln\sqrt{y^2 - x^2} + j\tanh^{-1}(x/y)$$

Conjugation of $z = x + jy$ by $\bar{z} = x - jy$.

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \& \qquad \bar{z}\,\bar{w} = \overline{zw}$$

Observe, if $z = x + jy$ and $z\bar{z} = x^2 - y^2 \neq 0$ then

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - jy}{x^2 - y^2}$$

The length of a hyperbolic number $\|z\| = \sqrt{x^2 + y^2} \neq \sqrt{z\bar{z}}$. In fact, the norm of the hyperbolic numbers is submultiplicative: $\|zw\| \leq \sqrt{2}\|z\|\|w\|$. This makes analysis on $\mathcal{H}$ more challenging than complex analysis.

Quadratic equations in $\mathcal{H}$ have surprising behavior; $z^2 = 1$ is solved by $z = \pm 1, \pm j$ thus

$$z^2 - 1 = (z - 1)(z + 1) = (z - j)(z + j)$$

factoring over $\mathcal{H}$ is not unique. This algebra has implications for the solution of differential equations over $\mathcal{H}$:
Example: $y'' - y = 0$ has characteristic equation $\lambda^2 - 1 = 0$ thus $\lambda = \pm 1, \pm j$ are characteristic values and $y = e^z, e^{-z}, e^{jz}, e^{-jz}$ are solutions. These are linearly dependent in $\mathcal{H}$ as:

$$e^{jz} = \frac{1}{2}(1 + j)e^z + \frac{1}{2}(1 - j)e^{-z} \;\; \& \;\; e^{-jz} = \frac{1}{2}(1 + j)e^{-z} + \frac{1}{2}(1 - j)e^z.$$

Hyperbolic Calculus

Differentiability over $\mathcal{H}$:
Let $U \subseteq \mathcal{H}$ be an open set containing p. If $f : U \to \mathcal{H}$ is a function then we say f is **$\mathcal{H}$-differentiable at** p if there exists a linear function $d_pf : \mathcal{H} \to \mathcal{H}$ such that $d_pf(vw) = d_pf(v)w$ for all $v, w \in \mathcal{H}$ and

$$\lim_{h \to 0} \frac{f(p + h) - f(p) - d_pf(h)}{||h||} = 0. \qquad (1)$$

The condition $d_pf(vw) = d_pf(v)w$ is called **$\mathcal{H}$-linearity**. We define the **hyperbolic derivative** by $f'(p) = df_p(1)$. Alternatively, we can define the derivative by a *deleted difference quotient:*

$$f'(p) = \lim_{\mathcal{H}^\times \ni z \to p} \frac{f(z) - f(p)}{z - p}$$

Since $z = x + jy$ in $\mathcal{H}$ we have $\{1, j\}$ as the basis for $\mathcal{H}$. Advanced calculus gives $d_pf(1) = (\partial_x f)(p)$ whereas $d_pf(j) = (\partial_y f)(p)$. Thus, as $d_pf(1) = d_pf(jj) = d_pf(j)j$,

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = j\frac{\partial f}{\partial y}.$$

If $f = u + jv$ then $\partial_y f = (\partial_y f)j$ yields **$\mathcal{H}$-Cauchy-Riemann Eqns:**

$$u_x = v_y \;\; \& \;\; u_y = v_x \;\; \Rightarrow \;\; J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & v_x \\ v_x & u_x \end{bmatrix}.$$

The Jacobian of an $\mathcal{H}$-differentiable function is in the regular representation of $\mathcal{H}$. Indeed, $J_f(p) = \mathbf{M}(f'(p))$.

Laplace-like equation for $\mathcal{H}$:
In complex analysis the CR-eqns for $f = u + iv$ are given by $f_y = if_x$. Then $f_{yy} = (f_y)_y = (if_x)_y = i(f_y)_x = i^2 f_{xx}$ then as $i^2 = -1$ we obtain the **Laplace Equation** $f_{xx} + f_{yy} = 0$. In contrast, for $\mathcal{H}$-differentiable $f = u + jv$ we derive

$$f_{yy} = (f_y)_y = (jf_x)_y = j(f_y)_x = j^2 f_{xx} \;\; \Rightarrow \;\; f_{yy} = f_{xx}$$

This is a **speed one wave equation** (think of y as time and x as space).

> If $f = u + jv$ then both u and v are solutions of the wave equation. Component functions of $\mathcal{H}$-differentiable function solve the wave equation.

Speed c Wave Equation:
Consider the wave equation $c^2u_{xx} = u_{tt}$. Define:

$$\mathcal{W}_c = \{x + kt \mid x, t \in \mathbb{R}, k^2 = c^2\}$$

Observe $\mathcal{W}_c$-differentiable f solve the speed-c wave equation:

$$k^2 = c^2 \;\; \Rightarrow \;\; c^2u_{xx} = u_{tt}. \qquad (2)$$

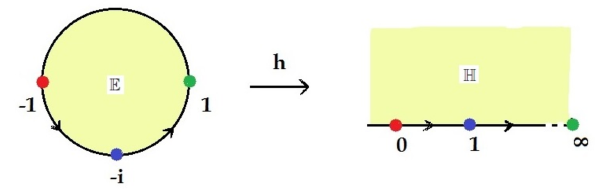
Since $\mathcal{W}_c$ is isomorphic to $\mathbb{R} \times \mathbb{R}$ by $\Phi(x + kt) = (x + ct, x - ct)$ we can trade each $\mathcal{W}_c$-differentiable function $f : \mathcal{W}_c \to \mathcal{W}_c$ for a corresponding with a $\mathbb{R} \times \mathbb{R}$ differentiable function F via $f = \Phi^{-1} \circ F \circ \Phi$ where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. The structure of $\mathbb{R} \times \mathbb{R}$-differentiable functions is rather simple; $F(a, b) = (F_1(a), F_2(b))$ where F_1, F_2 are differentiable functions on $\mathbb{R}$. Thus,

$$\begin{aligned} f(x + kt) &= \Phi^{-1}(F(\Phi(x + kt))) \qquad (3) \\ &= \Phi^{-1}(F_1(x + ct), F_2(x - ct)) \\ &= \frac{1}{2}(F_1(x + ct) + F_2(x - ct)) + \frac{k}{2c}(F_1(x + ct) - F_2(x - ct)) \end{aligned}$$

The above is the well-known d'Alembert's solution to the wave-equation.

Non-Conformal Mapping

Conformal Mapping:
In complex analysis the technique of conformal mapping allows us to find interesting solution of Laplace's Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. For example, the map $f(z) = i\left(\frac{z+1}{1-z}\right)$ can be visualized by:



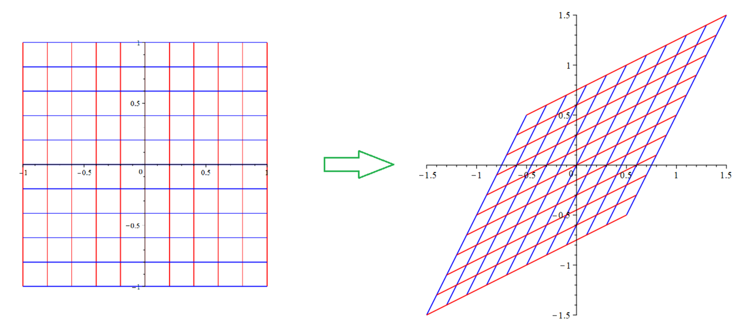
Laplace's Equation for the half-plane is with $w = u + iv$ is solved by

$$\Psi(w) = \frac{1}{\pi}\cot^{-1}(v/u)$$

Notice, if $w = u + iv = u$ we find $\Psi = 1$ for $u < 0$ and $\Psi = 0$ for $u > 0$. Then $\phi(z) = \Psi(f(z))$ gives a solution of Laplace's Equation on the disk where $\phi = 1$ on the upper semi-circle whereas $\phi = 0$ on the lower semi-circle.

- the term **conformal** means *angle-preserving*

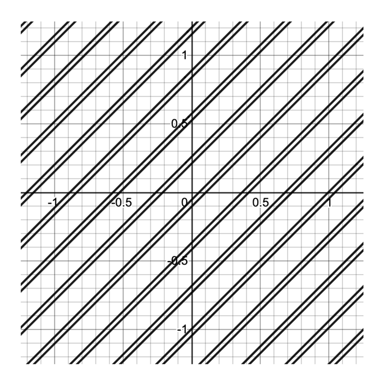
Non-Conformal Mapping:
In hyperbolic calculus we can also study a hyperbolic differentiable map given by $w = f(z)$. If $f : S \to T$ and we solve the **wave-equation** on T subject to appropriate boundary conditions by $\Psi = \Psi(w)$ then $\phi(z) = \Psi(f(z))$ defines a solution of the wave equation on S with matching boundary conditions. This is the hyperbolic analog of conformal mapping. However, we should caution, hyperbolic maps do not preserve the angle between curves. For instance: $f(z) = \left(\frac{1}{2} + j\right)z$ maps the orthogonal grid on the left to the slanted grid on the right:

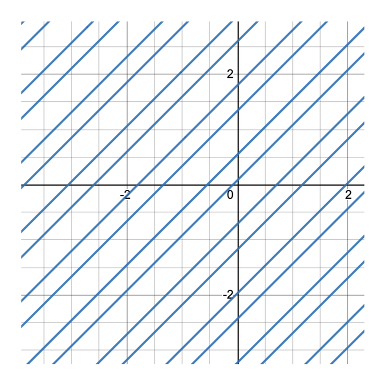


Strategy:

- Study component functions of hyperbolic differentiable maps to find solutions of the wave equation. Level curves of the component function are a convenient visual guide for the wide variety of functions possible in the function theory of $\mathcal{H}$. These will form *template* solutions for our $\mathcal{H}$-conformal mapping technique,
- (Future Work) Study fractional linear transformations on $\mathcal{H}$ as well as other maps to understand standard methods for the transport of curves on $\mathcal{H}$.

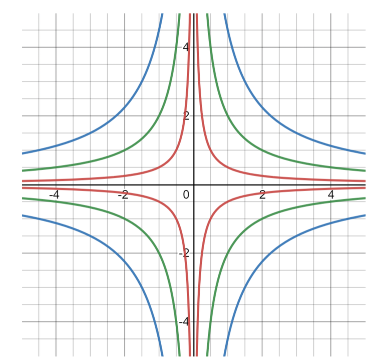
Wave Functions:
If $f(z) = Asin(kx - wt) - jAsin(kx - wt)$ then the level curves of the component functions $f(z) = Asin(kx - wt)$ and $-Asin(kx - wt)$. Now what you can notice here is when rewritten a curiosity appears: $A(1 - j)sin(kx - wt)$. Visually:

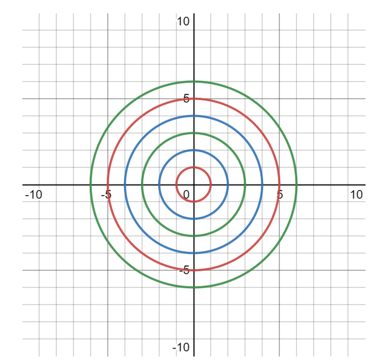




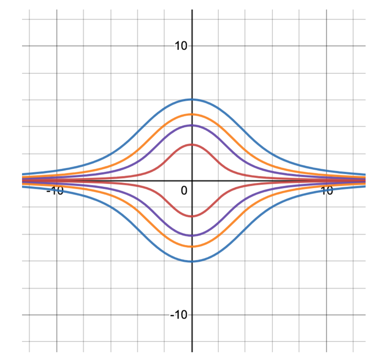
Level Curves of Hyperbolic Maps

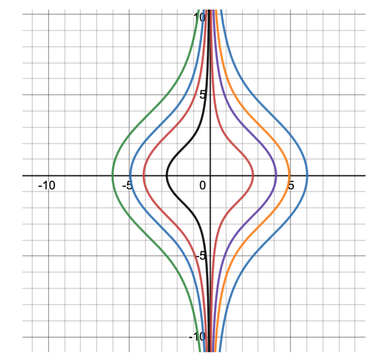
Square Function:
If $z^2 = x^2 + y^2 + 2jxy$ then the level curves of the component functions of $f(z) = z^2$ are $x^2 + y^2 = c_1$ and $2xy = c_1$. Visually:



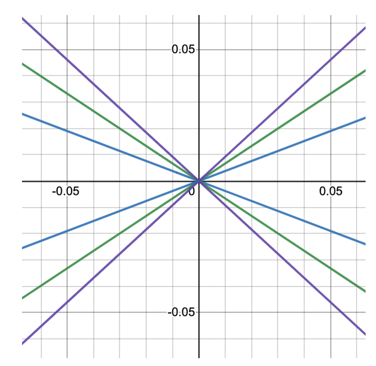


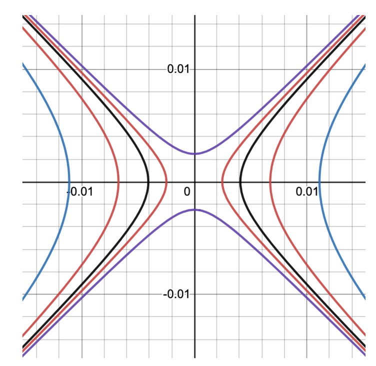
Cube Function:
If $f(z) = z^3 = x^3 + 3xy^2 + j(3x^2y + y^3)$ then the level curves of the component functions $f(z) = z^3$ are $z^3 = x^3 + 3xy^2 = c_1$ and $3x^2y + y^3 = c_2$. Visually:



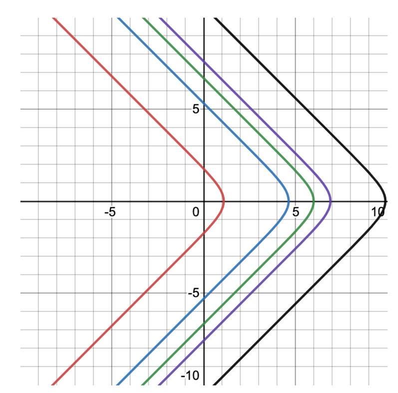


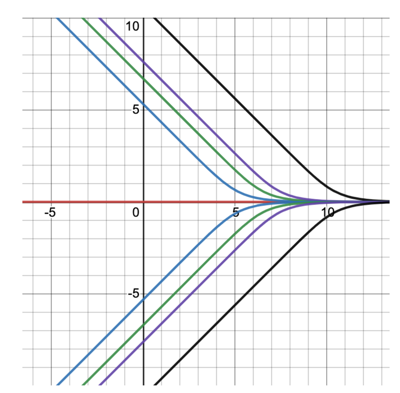
Log Function:
If $f(z) = Log(z) = Log(x^2 - y^2) + jtanh^-1(y/x)$ then the level curves of the component functions $f(z) = Log(z)$ are $Log(z) = Log(x^2 - y^2)$ and $(tanh^-1)(y/x)$. Visually:





Exponential Function:
If $f(z) = e^z = e^x(cosh(y) + jsinh(y))$ then the level curves of the component functions $f(z) = e^z$ are $e^z = e^x cosh(y)$ and $e^x sinh(y)$ Visually:





More Wave Functions:
When analyzing wave functions more carefully we discover that certain patterns arise naturally from solutions to the wave equation. $f(z) = sin(kx)cos(wt) + jcos(kx)sin(wt)$ is a solution in which fascinating geometries arise from studying the level curves $sin(kx)cos(wt)$ and $cos(kx)sin(wt)$ respectfully. Visually: